

Fidelity susceptibility for Lifshitz geometries and its application to non-relativistic many-body systems

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Abstract

In order to analyze the fidelity susceptibility of non-relativistic field theories, which are important in condensed matter systems, we generalize the proposal to obtain the fidelity susceptibility holographically to Lifshitz geometries. We use this proposal to analyze the fidelity susceptibility for a non-relativistic many-body system, and argue that the fidelity susceptibility of this theory can be holographically obtained from a bulk Lifshitz geometry. In fact, using a Einstein-Dilaton-Maxwell-AdS-Lifshitz theory, we explicitly demonstrated that the fidelity susceptibility obtained from this bulk geometry is equal to the fidelity susceptibility of a bosonic many-body system.

1 Introduction

It is known that the entropy of a black hole scales with its area. As black holes are maximum entropy objects, this implies that the maximum entropy of that certain region of space scales with the area of its boundary. This observation has led to the development of the holographic principle, which equates the number of degrees of freedom in a region of space to the number of degrees of freedom on the boundary surrounding that region of space [1, 2]. The AdS/CFT correspondence is a realization of the holographic principle as it is a duality between the string theory/supergravity in AdS spacetime and the field theory on its boundary [3]. As AdS/CFT correspondence is a duality between two very different theories, it seems from the AdS/CFT correspondence and the holographic principle that

laws of physics are fundamentally just information theoretical processes. In fact, various studies done in different fields of science seem to indicate that the laws of physics are informational theoretical processes [4, 5]. So, the AdS/CFT can be used to obtain information theoretical information relating to a conformal field theory from the bulk geometry. The entanglement entropy of a field theory is a most important informational theoretical quantity relating to a conformal field theory. It has been demonstrated that the entanglement entropy of a conformal field theory can be holographically obtained from the bulk AdS spacetime, as it is dual a minimal surface in asymptotically AdS spacetime [6, 7].

It is also important to know how much information is retained in a system, and holographic entanglement entropy can be used to quantify this as it measures the loss of information in a system. However, it is also important to know the difficulty to obtain this information, and this can be quantified using complexity. As laws of physics can be understood in terms of information theoretical processes [4, 5], and complexity is an important informational theoretical quantity, complexity is expected to be an important physical quantity used in the laws of physics. In fact, complexity has been used to understand the behavior of condensed matter systems [8, 9] and molecular physics [10], quantum computing [11]. In fact, it has been argued that the information might not be ideally lost in a black hole, but it would be effectively lost, as it would not be possible to obtain this information from a black hole due to its chaotic nature [12].

The complexity of a conformal field theory can also be obtained holographically, as the holographic complexity is dual to a volume in AdS spacetime [13, 14, 15, 16, 17]. It has been demonstrated that the holographic complexity of a field theory can be related to the fidelity susceptibility, and the fidelity susceptibility can be calculated using a maximal volume $V(\gamma_{max})$ in the AdS which ends on the time slice at the AdS boundary [18, 19]. As fidelity susceptibility is important to understand the behavior of condensed matter systems [20, 21, 22, 23, 24], it is important to generalize this proposal to non-relativistic field theories describing condensed matter systems. It may be noted that such non-relativistic condensed matter systems can be holographically analyzed using Lifshitz holography [25, 26, 27, 28, 29]. So, we will now generalize this proposal to calculate the fidelity susceptibility holographically [18, 19] to Lifshitz geometries. This can be done by first defining $V(\gamma_{max})$ as a maximal volume in the Lifshitz deformation of the AdS spacetime, which ends on the time slice at the Lifshitz-AdS boundary. Then using this maximal volume in the Lifshitz geometry, we can define the fidelity susceptibility of a non-relativistic boundary theory as

$$\Xi_F = \frac{V(\gamma_{max})}{8\pi R G}, \quad (1)$$

where R is the radius of the curvature of this Lifshitz-AdS geometry. It may be noted that for $z = 1$ this maximal volume reduces to the usual maximal volume in AdS spacetime, and so this fidelity susceptibility reduces to the usual fidelity susceptibility [18, 19] for $z = 1$.

Now we will use this proposal for holographically analyze a non-interacting system of N bosons, in a uniform magnetic field \vec{H} in the z direction, with $\vec{H} = H\hat{e}_z$. So, before we analyze the bulk Lifshitz geometry dual to such a system, we will calculate the fidelity susceptibility of this bosonic theory. The Hamiltonian for each of these bosonic particles is $H_i = (-i\vec{\nabla}_i - q\vec{A})^2/2m$,

where $\vec{A} = \vec{\nabla} \times \vec{H}$, so the time-dependent Schrödinger equation for these bosonic particles can be written as

$$\sum \frac{(-i\vec{\nabla}_i - q\vec{A})^2}{2m} \Psi_{tot}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N; t) = i \frac{\partial}{\partial t} \Psi_{tot}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N; t). \quad (2)$$

Now we can write the total wave function as $\Psi_{tot}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N; t) = \prod_{i=1}^N \psi_i(\vec{x}_i; t)$. We need to find only ground state wave function $\Psi_{tot}^0(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N; t)$. Choosing the gauge, $\vec{A} = (0, Hx, 0)$, we can write $\vec{A} = \vec{\nabla} \times \vec{H}$. Using this gauge for \vec{A} , Schrödinger equation can be expressed as

$$-\nabla_i^2 \psi_i(\vec{x}_i; t) + 2iqHx \frac{\partial \psi_i(\vec{x}_i; t)}{\partial y} + q^2 H^2 x^2 \psi_i(\vec{x}_i; t) = 2mE_i \psi_i(\vec{x}_i; t). \quad (3)$$

We can express $\psi_i(\vec{x}_i; t)$ as $\psi_i(\vec{x}_i; t) = e^{i(k_i z_i + \beta_i y_i - E_i t)} \phi_i(x_i)$, and obtain

$$-\frac{d^2 \phi_i(x_i)}{dx_i^2} + \left(q^2 H^2 x_i^2 - 2q\beta_i H x_i \right) \phi_i(x_i) = (2mE_i - k_i^2 - \beta_i^2) \phi_i(x_i). \quad (4)$$

It may be noted that this is the Schrödinger equation for a simple harmonic oscillator, such that the coordinates have been shifted as $x_i \rightarrow \xi + \frac{\beta_i}{qH}$. So, we can express the Schrödinger equation for this system as

$$-\frac{1}{2m} \frac{d^2 \phi_i(\xi_i)}{d\xi_i^2} + \frac{1}{2} m \omega_i^2 \xi_i^2 \phi_i(\xi_i) = \left(E_i - \frac{k_i^2}{2m} \right) \phi_i(\xi_i), \quad (5)$$

where $\omega_i = \frac{qH}{m}$ denotes frequency (energy) of the system. Exact solution for this equation can be expressed in terms of Hermite functions. Now using the ground state wave function for a bosonic particle, $\phi_{i,0}(\xi_i) = \sqrt[4]{\frac{qH}{\pi}} e^{-\frac{qH}{2} \xi_i^2}$, $E_{i,0} = \frac{qH + k_i^2}{2m}$, the ground state wave function for whole system can be written as

$$\Psi_{tot,0}(\vec{x}_1, \vec{x}_2, \dots, \vec{x}_N; t) = \left(\frac{qH}{\pi} \right)^{N/4} \prod_{i=1}^N e^{i \left(k_i z_i + \beta_i y_i - \left(\frac{qH + k_i^2}{2m} \right) t \right) - \frac{qH}{2} \left(x_i - \frac{\beta_i}{qH} \right)^2} \quad (6)$$

The fidelity susceptibility can be obtained by varying $H \rightarrow H + \delta H$, and computing the following inner product,

$$F = \langle \Psi_{tot,0}(H) | \Psi_{tot,0}(H + \delta H) \rangle = 1 - (\delta H)^2 \Xi_F + \dots$$

here we expand F in series up to the second order [30],

$$\begin{aligned} & \langle \Psi_{tot,0}(H) | \Psi_{tot,0}(H + \delta H) \rangle \\ &= \prod_{i=1}^N \int d^3 x_i \Psi_{tot,0}^*(H + \delta H) \Psi_{tot,0}(H) \\ &= \left(\int d^3 x \psi^*(\vec{x}; t, H + \delta H) \psi(\vec{x}; t, H) \right)^N. \end{aligned} \quad (7)$$

Thus, we can express the fidelity susceptibility of this system as

$$\Xi_F = \frac{N}{8qH^3} (qH + 4\beta^2). \quad (8)$$

Expression given in (8) is the exact fidelity susceptibility for a system of N charged bosonic particles in a uniform magnetic field.

This boundary theory is defined by non-relativistic Hamiltonian, so it is expected to be dual to a non-relativistic Lifshitz bulk theory [31, 32, 33, 34]. Now we will show that the Einstein-Dilaton-Maxwell-AdS-Lifshitz bulk action [35, 36, 37, 38] is holographically dual to the quantum non-relativistic bosonic system, and the fidelity susceptibility from the bulk theory matches the fidelity susceptibility of the boundary theory. Thus, we propose the following action for the bulk theory [37, 38],

$$S_{\text{Bulk}} = \int_{\mathcal{M}} d^4x \sqrt{-g} \left[\frac{(R - 2\Lambda)}{2\kappa^2} - \frac{1}{2} \partial^\mu \phi \partial_\mu \phi + V(\phi) - \xi e^{\lambda\phi} (F^{\mu\nu} F_{\mu\nu}) \right]. \quad (9)$$

where the potential is $V(\phi) = V_0 e^{\gamma\phi}$ with parameters V_0 and γ . Here ϕ is non-minimally coupled with electromagnetic potential, and the electromagnetic field strength coupled to scalar field as $\xi e^{\lambda\phi} (F^{\mu\nu} F_{\mu\nu})$, such that ξ, λ are suitable constants.

The metric for a static, spherically symmetric solution in this Einstein-Dilaton-Maxwell-AdS-Lifshitz can be written as [37, 38]

$$ds^2 = -e^{2\alpha(r)} B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 d\sigma_{2,k}^2, \quad (10)$$

where $\alpha(r), B(r)$ are function of r . Here $d\sigma_{2,k}^2$ is the metric for a topological two-dimensional surface parametrized by $k = 0, \pm 1$. This two-dimensional manifold is a sphere S_2 for $k = 1$, a torus T_2 for $k = 0$, and a compact hyperbolic manifold Y_2 for $k = -1$. Now we can choose $k = 0$ and write the planar Euclidean coordinates as $d\sigma_{2,k}^2 = dx^i dx_i$, $i = 1, 2, x^i = \{x, y\}$. It may be noted that due to Lifshitz scaling, $\alpha(r) \propto \log r^{z/2}$, where z is the Lifshitz parameter. So, the general form of the metric with Lifshitz symmetry can be written as

$$ds^2 = -\left(\frac{r}{r_0}\right)^z B(r) dt^2 + \frac{dr^2}{B(r)} + r^2 dx_i dx^i. \quad (11)$$

Here the function $B(r)$ can be written as [37, 38]

$$\begin{aligned} B(r) = & \frac{2}{(2+z)} \left[\frac{\tilde{V}_0}{2} \right] + \left(\frac{r_+}{r} \right)^{1+z/2} \left(\left(\frac{2(\Lambda + \tilde{Q}^2 \xi)}{(6+z)} \right) r_+^2 \right. \\ & \left. - \frac{2}{(2+z)} \left[\frac{\tilde{V}_0}{2} \right] \right) - \left(\frac{2(\Lambda + \tilde{Q}^2 \xi)}{(6+z)} \right) r^2. \end{aligned} \quad (12)$$

Now we will use generalization the fidelity susceptibility [18, 19] to a Lifshitz geometry given by Eq. (1). The fidelity susceptibility in such geometries depends on $V(\gamma_{max})$, and we can obtain $V(\gamma_{max})$ using

$$V(\gamma_{max}) = \int_{r_+}^{r_\infty} \frac{r^2 dr}{\sqrt{B(r)}} \quad (13)$$

where r_+ is horizon, and r_∞ is an IR cutoff. Now we can use the Poincare coordinate $w = \frac{r}{r_+}$ to evaluate integral (13) as

$$V(\gamma_{max}) = r_+^3 \int_\epsilon^1 \frac{dw}{w^4 \sqrt{b(w)}} \quad (14)$$

where $\epsilon \rightarrow 0$ is an *UV* cutoff. We also have

$$b(w) = b_0 + b_1 w^{1+z/2} - \frac{b_{-2}}{w^2}. \quad (15)$$

It may be noted as $z = -\frac{4\tilde{Q}^2\xi}{\Lambda + \tilde{Q}^2\xi} \geq 3$, $\Lambda = -\frac{3}{L^2}$, so $z = 4$ is an interesting solution. In this case, the coefficients b_n are given as following

$$b_0 = 0, \quad b_1 = \left(\frac{\Lambda + \tilde{Q}^2\xi}{5} \right) r_+^2 - \frac{1}{3} \left[\frac{\tilde{V}_0}{2} \right], \quad (16)$$

$$b_{-2} = \frac{r_+^2 (\Lambda + \tilde{Q}^2\xi)}{5}. \quad (17)$$

Now we obtain

$$V(\gamma_{max}) = r_+^3 \left(\frac{A}{3840(-b_{-2})^{9/2}} \right) + \frac{r_+^3}{2\epsilon^2 \sqrt{-b_{-2}}} \quad (18)$$

here $A = 640b_{-2}^3(b_1 - 3b_{-2})$. Here the bulk charge \tilde{Q} is dual to the magnetic charge (strength) H of the boundary theory, and H varying smoothly. So, we can the volume (18) for $\mathcal{O}(\frac{1}{H^3})$, and obtain,

$$\Xi_F = \frac{\sqrt{5}r_+^2\sqrt{-\xi}(2L^2\xi\tilde{Q} + 3)}{48\pi GL^3\xi^2\tilde{Q}^3} - \frac{\sqrt{5}r_+^2\sqrt{-\xi}(2L^2\xi\tilde{Q}^2 + 3)}{32\pi GL^3\xi^2\tilde{Q}^3r^2\epsilon^2}. \quad (19)$$

It may be noted that the finite part of (19) is same as (8). Thus, we can write

$$\frac{\sqrt{5}r_+^2\sqrt{-\xi}(2L^2\xi\tilde{Q} + 3)}{48\pi GL^3\xi^2\tilde{Q}^3} = \frac{N}{8qH^3}(qH + 4\beta^2) \quad (20)$$

Now if we holographically identify $\tilde{Q} = H$, we can also identify many parameters in the bulk to boundary theories. In fact, from this identification, we obtain

$$\frac{2\sqrt{5}L^2\xi r_+^2\sqrt{-\xi}}{48\pi GL^3\xi^2} \equiv \frac{N}{8}, \quad (21)$$

$$\frac{3\sqrt{5}r_+^2\sqrt{-\xi}}{48\pi GL^3\xi^2} = \frac{N\beta^2}{2q}. \quad (22)$$

So, the number of boundary quantum systems N and $\beta^2 q^{-1}$ can be expressed as

$$N^{\text{boundary}} = \left(-\frac{16\sqrt{5}L^2r_+^2}{48\pi GL^3\sqrt{-\xi}} \right)^{\text{Bulk}}, \quad (23)$$

$$\left(\frac{\beta^2}{q} \right)^{\text{boundary}} = \left(\frac{3}{8L^2\xi} \right)^{\text{Bulk}}. \quad (24)$$

It may be noted that $\xi < 0$. So, we have analyzed a system of non-interacting bosonic particles in a magnetic field. We obtained the fidelity susceptibility for this system. As this system was a non-relativistic system, it was expected to be dual to a AdS-Lifshitz spacetime. In fact, we demonstrated that this theory is

dual to a Einstein-Dilaton-Maxwell-AdS-Lifshitz, and the fidelity susceptibility calculated from the bulk using this theory matches with the fidelity susceptibility of the boundary theory.

As we have generalized the fidelity susceptibility to Lifshitz geometries, and Lifshitz geometries can have important condensed matter applications [25, 26, 27, 28, 29], the results of this paper can have interesting condensed matter applications. So, it would be interesting to analyze realistic condensed matter systems, and then use the Lifshitz holography to understand the behavior of fidelity susceptibility for such condensed matter systems. It is also possible to study various interesting time-dependent generalizations of this solution. It is expected that such a time-dependent system on the boundary will be dual to some time-dependent Lifshitz bulk solution. It may be noted that fidelity susceptibility for time-dependent geometries has been studied [39], and it is expected that this formalism can be generalized to bulk geometries with Lifshitz symmetry. It would be interesting to analyze such geometries, and use them to understand the boundary fidelity susceptibility.

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